Learning
Skills
Unit
RMIT
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Logarithms

Definition The **logarithm** of a number to a particular base is the **power (or index)** to which that **base** must be raised to obtain the number.

This means that a **logarithm** is an **index.**

Index or logarithm

The number 8 written in **index form** is $\mathbf{8 = 2^3}$

base

The equation can be rewritten in **logarithm form** as $\mathbf{\log_2 8 = 3}$

base

Index or logarithm

The logarithm statement reads

the logarithm of 8, to the base 2 is 3

and is equivalent to the index statement

8 equals 2 to the power 3
or 2 to the power 3 equals 8.

In general

$\mathbf{a}^x = \mathbf{n} \quad \Longleftrightarrow \quad \mathbf{\log_a n} = x$ (follow the arrows and read $\mathbf{a}^x = \mathbf{n}$) $(a > 0)$

Examples **1**. Change the following from index form to logarithm form.

(a) $10^3 = 1000$

using the expression above $\mathbf{a = 10,}$ $\boldsymbol{x}$ $\mathbf{=3,}$ $\mathbf{n = 1000}$ *so*

$$10^3 = 1000 \quad \Rightarrow \quad \log_{10} 1000 = 3$$

(b) $2^{-5} = \frac{1}{32}$

using the expression above $\mathbf{a = 2,}$ $\boldsymbol{x} \mathbf{= -5,}$ $\mathbf{n = \frac{1}{32}}$ *so*

$$2^{-5} = \frac{1}{32} \quad \Rightarrow \quad \log_2 \frac{1}{32} = -5$$

(c) $16^{\frac{1}{2}} = 4$

using the expression above **a = 16, *x* = $\frac{1}{2}$, n = 4**

$16^{\frac{1}{2}} = 4 \quad \Rightarrow \quad \log_{16} 4 = \frac{1}{2}$

2. Change the following from logarithm form to index form .

(a) $\log_2 16 = 4$

the base is 2, *the number is* 16, *the index is* 4

so $\log_2 16 = 4 \quad \Rightarrow \quad 2^4 = 16$ (*follow the arrows and read* $2^4 = 16$)

(b) $\log_{10} 10 = 1$

the base is 10, *the number is* 10, *the index is* 1

so $\log_{10} 10 = 1 \Rightarrow 10^1 = 10$

(c) $\log_3 \left(\frac{1}{81}\right) = -4$

the base is 3, *the number* is $\frac{1}{81}$, *the index is* -4

$$\log_3 \left(\frac{1}{81}\right) = -4 \quad \Rightarrow \quad 3^{-4} = \frac{1}{81}$$

Note **1.** $\log_a 1 = 0$ The logarithm of 1 to any base is 0.

2 $\log_a a = 1$ The logarithm of a number to the same base is 1.

Exercise 1 Write in logarithm form

(a) $3^2 = 9$ (b) $10^4 = 10\ 000$

(c) $64^{\frac{1}{2}} = 8$ (d) $10^{-2} = 0.01$

(e) $x^5 = y$ (f) $z^0 = 1$

Exercise 2 Write in index form

(a) $\log_{10} 100 = 2$ (b) $\log_2 32 = 5$

(c) $\log_{10} 0.001 = -3$ (d) $\log_4 x = 3$

(e) $\log_3 9 = x$ (f) $\log_a a = 1$

Evaluating logarithms To evaluate a logarithm:

1. write the number in index form, with same base as the logarithm
2. use the definition of a logarithm to evaluate.

Examples (1) Evaluate $\log_5 125$

$125 = 5^3$ 125 *in index form with base* 5. *The logarithm is* 3.
$\therefore \quad \log_5 125 = 3$ *equivalent logarithm statement*

(2) Evaluate $\log_{10} 0.0001$

$0.0001 = 10^{-4}$ 0.0001 *in index form with base 10. The logarithm is* – 4.
$\therefore \quad \log_{10} 0.0001 = -4$ *equivalent logarithm statement*

Exercise 3 Without using a calculator, evaluate the following:

(a) $\log_7 49$ (b) $\log_{10} \sqrt{10}$

(c) $\log_2 128$ (d) $\log_{10} 100\ 000$

(e) $\log_5 1$ (f) $\log_3 \dfrac{1}{27}$

(g) $\log_{10} 0.001$

Solving equations (1)

Equations of the type $\log_a x = b$ and $\log_x a = b$ can be solved for x by rewriting the equation in index form.

Examples

Solve the following for x.

(1) $\log_2 x = 5$

$$\log_2 x = 5 \;\Rightarrow\; 2^5 = x$$

therefore $x = 32$

(2) $\log_x 64 = 3$

$$\log_x 64 = 3 \;\Rightarrow\; x^3 = 64$$

$$x^3 = 4^3 \quad \textit{equate bases}$$

therefore $x = 4$

Exercise 4

Solve the following equations for the unknown

(a) $\log_3 x = 4$

(b) $\log_{10} x = -3$

(c) $\log_a 4 = \frac{1}{2}$

(d) $\log_y 36 = 2$

(e) $\log_x 27 = 3$

(f) $\log_x\left(\frac{1}{9}\right) = \frac{-2}{3}$

Logarithm laws

The logarithm laws are obtained from the index laws and are:

$\log_a x + \log_a y = \log_a xy$	*eg.* $\log_2 5 + \log_2 3 = \log_2 (5\times 3)$ $= \log_2 15$
$\log_a x - \log_a y = \log_a \frac{x}{y}$	*eg.* $\log_{10} 12 - \log_{10} 3 = \log_{10} \frac{12}{3}$ $= \log_2 4$
$\log_a x^y = y \log_a x$	*eg.* $\log_3 7^4 = 4 \log_3 7$
$\log_a \frac{1}{x} = -\log_a x$	*eg.* $\log_a \frac{1}{10} = -\log_a 10$
$\log_a 1 = 0$	*eg.* $\log_{10} 1 = 0$
$\log_a a = 1$	*eg.* $\log_5 5 = 1$
$a^{\log_a x} = x$	*eg.* $10^{\log_{10} 4} = 4$

Note: *It is not possible to have the logarithm of a negative number.*
Only logarithms with the same base can be simplified using log laws.

Examples Express the following as a single logarithm.
(Remember the logarithms must have the same base if they are to be added or subtracted).

(1) $\log_3 5+\log_3 20-\log_3 10$

Use the laws for adding and subtracting logarithms.

$$\begin{aligned}&\log_3 5+\log_3 20-\log_3 10\\ &=\log_3 (5\times 20)-\log_3 10\\ &=\log_3\left(\frac{5\times 20}{10}\right)\\ &=\log_3 10\end{aligned}$$

(2) $\log_2 5+3\log_2 3-2\log_2 6$

$3\log_2 3$ *and* $2\log_2 6$ ***must*** *be written as* $\log_2 3^3$ *and* $\log_2 6^2$ ***before*** *using addition and subtraction laws.*

$$\begin{aligned}&\log_2 5+3\log_2 3-2\log_2 6\\ &=\log_2 5+\log_2 3^3-\log_2 6^2\\ &=\log_2\left(\frac{5\times 3^3}{6^2}\right)\\ &=\log_2\left(\frac{15}{4}\right)\end{aligned}$$

(3) Simplify and evaluate

$$\frac{1}{2}\log_{10} 36-\log_{10} 15+2\log_{10} 5$$

$=\log_{10} 36^{\frac{1}{2}}-\log_{10} 15+\log_{10} 5^2$ *write each term in the form* $\log_{10}(\ )$

$=\log_{10} 6-\log_{10} 15+\log_{10} 25$ *use laws for adding and subtracting logarithms*

$$\begin{aligned}&=\log_{10}\frac{6\times 25}{15}\\ &=\log_{10} 10\\ &=1\end{aligned}$$

Exercise 5 Express as a single logarithm and evaluate, if possible, without using a calculator .

(a) $\log_4 8 + \log_4 3 - \log_4 2$

(b) $\log_2 5 + 2\log_2 4 + \log_2 \dfrac{2}{5}$

(c) $\dfrac{1}{2}\log_{10} 50 - \log_{10} 4 + 2\log_{10} 3$

(d) $\log_{10} 40 - \log_{10} 15 + 2\log_{10}\left(\dfrac{3}{5}\right)$

(e) $\log_a 4 + 2\log_a 3 - 2\log_a 6$

(f) $\log_e 2e^3 + 2\log_e 3 - \log_e 18$

Solving equations (2) Logarithms can be used to solve indicial equations of the form $a^x = b$.

We solve these equations by taking logarithms of both sides. By taking the logarithms to the base 10 or the base e (Euler's number), we can use a calculator to evaluate logarithms. Logarithms to the base e are often called *natural logarithms.*

On the calculator use the **log** button to evaluate logarithms to the base 10 and the **ln** button to evaluate logarithms to the base e.

Examples Solve for *x* giving your answer to three decimal places.

(a) $3^x = 7$

$\log 3^x = \log 7$ *Take logarithms of both sides.*

$x \log 3 = \log 7$ *Use* $\log_a x^y = y \log_a x$

$$x = \frac{\log 7}{\log 3}$$

$$x = 1.771$$

(b) $2 \times 5^{x+1} = 15$

$5^{x+1} = 7.5$ *Divide both sides by 2*

$\ln 5^{x+1} = \ln 7.5$ *Take logarithms of both sides*

$(x+1)\ln 5 = \ln 7.5$ *Use* $\ln x^y = y \ln x$

$$(x+1) = \frac{\ln 7.5}{\ln 5} = 1.252$$

$$x = 0.252$$

(c) $6^{x-1} = 3^x$

$\log 6^{x-1} = \log 3^x$ *Take logarithms of both sides*

$(x-1)\log 6 = x\log 3$ *Use* $\log_a x^y = y\log_a x$

$x\log 6 - \log 6 = x\log 3$

$x(\log 6 - \log 3) = \log 6$

$$x = \frac{\log 6 - \log 3}{\log 6}$$

$x = 0.387$

Exercise 6 Solve for the unknown, giving your answer to three decimal places.

(a) $5^x = 12$

(b) $2^{x-3} = 9$

(c) $3 \times e^{-\frac{x}{2}} = 0.015$

(d) $\left(\frac{1}{2}\right)^x - 1 = 2.5$

(e) $4^{x+1} = 5^x$

(f) $2\left(3^{1-2x}\right) - 1 = 3$

Answers

Exercise 1

(a) $\log_3 9 = 2$ (b) $\log_{10} 10000 = 4$ (c) $\log_{64} 8 = \frac{1}{2}$ (d) $\log_{10} 0.01 = -2$

(e) $\log_x y = 5$ (f) $\log_z 1 = 0$

Exercise 2

(a) $10^2 = 100$ (b) $2^5 = 32$ (c) $10^{-3} = 0.001$ (d) $4^3 = x$

(e) $3^x = 9$ (f) $a^1 = a$

Exercise 3

(a) 2 (b) $\frac{1}{2}$ (c) 7 (d) 5

(e) 0 (f) – 3 (g) – 3

Exercise 4

(a) $x = 81$ (b) $x = 0.001$ (c) $a = 16$ (d) $y = 6$

(e) $x = 3$ (f) $x = 27$

Exercise 5

(a) $\log_4 12$ (b) 5 (c) $\log_{10} \frac{45\sqrt{2}}{4}$ (d) $\log_{10} \frac{24}{25}$

(e) 0 (f) 3

Exercise 6

(a) $x = 1.544$ (b) $x = 6.170$ (c) $x = 10.597$ (d) $x = -1.807$

(e) $x = 6.21$ (f) $x = 0.185$