In your study of Differential Equations so far you have probably solved first and second order equations using methods such as separation of variables, substitutions, homogeneous equations, and integrating factor technique.

Laplace transforms are another means of solving some differential equations that may prove too difficult to solve using the above mentioned methods.

*The use of partial fractions is a fundamental component of Laplace transforms, and you should revise partial fractions thoroughly before starting Laplace transforms.*

The fundamental rule for Laplace Transforms is:

\[
L[y(t)] = Y(s) = \int_{0}^{\infty} e^{-st} y(t) \, dt
\]

Rather than perform what can be a complicated integration, a table is provided with some of the most common transforms already completed:

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y(t) )</td>
<td>( L[y(t)]=Y(s) )</td>
<td>(i)</td>
</tr>
<tr>
<td>1</td>
<td>( L[1]=\frac{1}{s} )</td>
<td>(ii)</td>
</tr>
<tr>
<td>( t^n )</td>
<td>( L[t^n]=\frac{n!}{s^{n+1}} )</td>
<td>(iii)</td>
</tr>
<tr>
<td>( e^{at} )</td>
<td>( L[e^{at}]=\frac{1}{s-a} )</td>
<td>(iv)</td>
</tr>
<tr>
<td>( \sin kt )</td>
<td>( L[\sin kt]=\frac{k}{s^2 + k^2} )</td>
<td>(v)</td>
</tr>
<tr>
<td>( \cos kt )</td>
<td>( L[\cos kt]=\frac{s}{s^2 + k^2} )</td>
<td>(vi)</td>
</tr>
<tr>
<td>( \sinh kt )</td>
<td>( L[\sinh kt]=\frac{k}{s^2 - k^2} )</td>
<td>(vii)</td>
</tr>
<tr>
<td>( \cosh kt )</td>
<td>( L[\cosh kt]=\frac{s}{s^2 - k^2} )</td>
<td>(viii)</td>
</tr>
<tr>
<td>( \frac{dy}{dt} )</td>
<td>( L\left[\frac{dy}{dt}\right]=sY(s)-y(0) )</td>
<td>(ix)</td>
</tr>
<tr>
<td>( \frac{d^2 y}{dt^2} )</td>
<td>( L\left[\frac{d^2 y}{dt^2}\right]=s^2Y(s)-sy(0)-y'(0) )</td>
<td>(x)</td>
</tr>
</tbody>
</table>
e.g. 1

Given the following first order differential equation, \( \frac{dy}{dt} + y = 3e^{2t} \), where \( y(0) = 4 \).

Find \( y(t) \) using Laplace Transforms.

So

To begin solving the differential equation we would start by taking the Laplace transform of both sides of the equation.

\[
L \left[ \frac{dy}{dt} + y \right] = L[3e^{2t}]
\]

Taking the Laplace Transform of both sides of the equation.

\[
L \left[ \frac{dy}{dt} \right] + L[y] = 3L[e^{2t}]
\]

Separating terms.

\[
sY(s) - y(0) + Y(s) = 3 \times \frac{1}{s-2}
\]

Transforms as derived from tables.

\[
sY(s) - 4 + Y(s) = \frac{3}{s-2}
\]

Substituting for \( y(0) = 4 \)

\[
Y(s)(s+1) = \frac{3}{s-2} + 4
\]

Taking \( Y(s) \) as a common factor.

\[
Y(s) = \frac{3}{(s-2)(s+1)} + \frac{4}{s+1}
\]

\[
Y(s) = \frac{3}{(s+1)(s-2)} + \frac{4(s-2)}{(s+1)(s-2)}
\]

\[
Y(s) = \frac{4s-5}{(s-2)(s+1)}
\]

Making \( Y(s) \) the subject.

Use partial fractions to expand \( \frac{4s-5}{(s-2)(s+1)} \)

\[
\therefore \quad \frac{4s-5}{(s-2)(s+1)} = \frac{A}{s-2} + \frac{B}{s+1}
\]

\[
4s-5 = A(s+1) + B(s-2)
\]

By selecting appropriate values of \( s \), we can solve for \( A \) & \( B \).

Letting \( s = -1 \), and substituting into the above equation gives
\[4(-1) - 5 = A(-1+1) + B(-1 - 2)\]
\[-4 - 5 = A(0) + B(-3)\]
\[-9 = -3B\]
\[B = \frac{-9}{-3} = 3\]

Now let \(s = 2\), and substitute into the same equation

\[4(2) - 5 = A(2+1) + B(2 - 2)\]
\[8 - 5 = A(3) + B(0)\]
\[3 = 3A\]
\[A = \frac{3}{3} = 1\]

So

\[\frac{4s - 5}{(s-2)(s+1)} = \frac{1}{s-2} + \frac{3}{s+1}\]

Therefore

\[Y(s) = \frac{1}{s-2} + \frac{3}{s+1}\]

To obtain a solution \(y(t)\) to the differential equation from \(Y(s)\) we need to find the inverse Laplace transform of \(Y(s)\).

\[\therefore \quad L^{-1}[Y(s)] = L^{-1}\left[\frac{1}{s-2} + \frac{3}{s+1}\right]\]

\[\therefore \quad y(t) = L^{-1}\left[\frac{1}{s-2}\right] + 3L^{-1}\left[\frac{1}{s+1}\right]\]

\[\therefore \quad y(t) = e^{2t} + 3e^{-t}\] 
Inverse transforms obtained from tables.
Given the following first order differential equation, \( y'' + y' = 3 \cos 2t; \ y(0) = 0; \ y'(0) = 0 \)

Find \( y(t) \) using Laplace Transforms.

\[
L[y''] + L[y'] = 3L[\cos 2t]
\]

\[
s^2Y(s) - sY(s) = \frac{3s}{s^2 + 2^2}
\]

\[
Y(s)(s^2 + 1) = \frac{3s}{s^2 + 4}
\]

\[
Y(s) = \frac{3s}{(s^2 + 4)(s^2 + 1)}
\]

\[
\frac{3s}{(s^2 + 4)(s^2 + 1)} = \frac{As + B}{s^2 + 4} + \frac{Cs + D}{s^2 + 1}
\]

\[
\therefore 3s = (As + B)(s^2 + 1) + (Cs + D)(s^2 + 4)
\]

An alternative method for solving the unknowns \( A, B, C, \) & \( D \) in the above equation is called “equating coefficients of powers of \( s \)”. Rewriting the equation as:

\[
0s^3 + 0s^2 + 3s^1 + 0s^0 = (As + B)(s^2 + 1) + (Cs + D)(s^2 + 4)
\]

\[
LHS = RHS
\]

\[
s^0 : \quad 0 = B + 4D \quad \text{eq2 1.}
\]

\[
s^1 : \quad 3 = A + 4B \quad \text{eq2 2.}
\]

\[
s^2 : \quad 0 = B + D \quad \text{eq2 3.}
\]

\[
s^3 : \quad 0 = A + C \quad \text{eq2 4.}
\]

Solving the simultaneous equations for \( A, B, C, \) & \( D \) gives:

\[
B = -D \quad \text{Rearrange eq^2 3.} \quad A = -C \quad \text{Rearrange eq^2 4.}
\]

\[
0 = -D + 4D \quad \text{Substitute into eq^2 1.} \quad 3 = -C + 4C \quad \text{Substitute into eq^2 2.}
\]

\[
0 = 3D \quad \text{Solve for } D. \quad C = 1 \quad \text{Solve for } C.
\]

\[
D = 0
\]

\[
\therefore A = -1, \quad B = 0, \quad C = 1, \quad D = 0
\]

\[
\therefore Y(s) = \frac{-s}{s^2 + 4} + \frac{s}{s^2 + 1}
\]

\[
\therefore y(t) = L^{-1}[Y(s)] = L^{-1}\left[\frac{-s}{s^2 + 4}\right] + L^{-1}\left[\frac{s}{s^2 + 1}\right] \quad \text{using tables}
\]

\[
y(t) = -\cos 2t + cost
\]